Magnetization Curves, Critical Exponents, and Hybrid-Order Finite-Temperature Phase Transition of the d=1 Long-Range Ising Model

E. Can Artun[1,2] and A. Nihat Berker[3,2,4]

[1] *TUBITAK Research Institute for Fundamental Sciences (TBAE), Gebze, Kocaeli 41470, Turkey*
[2] *Faculty of Engineering and Natural Sciences, Kadir Has University, Cibali, Istanbul 34083, Turkey*
[3] *Faculty of Engineering, MEF University, Maslak, Istanbul 34396, Turkey*
[4] *Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

The $d = 1$ Ising ferromagnet, antiferromagnet, and spin glass with long-range power-law interactions $J\,r^{-a}$ are studied for all interaction range exponents a by a renormalization-group transformation that simultaneously projects local ferromagnetism and antiferromagnetism. Phase diagrams in temperature and interaction-range exponent a, critical exponents and magnetization curves as a function of temperature for different a are calculated. In the ferromagnetic case, $J > 0$, a finite-temperature ferromagnetic phase occurs for interaction range $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, the phase transition becomes hybrid order, namely with discontinuous magnetization and essential singularity in the specific heat, as predicted by rigorous results. For $a > 2$, the phase transition temperature discontinuously drops to zero and for $a > 2$ there is no ordered phase above zero temperature, also as predicted by rigorous results. At the other end, on approaching $a = 0.74$ from above, namely increasing the range of the interaction, the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before ($a > 0$) the neighbors become equivalent, namely before the interactions become equal for all separations. The critical exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$ are calculated, from a large 20×20 recursion matrix determined at the unstable fixed points, and continuously vary as a function of a. For the antiferromagnetic case, $J < 0$, all triplets of spins at all ranges have competing interactions and our calculation shows that this highly frustrated system does not have an ordered phase. In the spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p), a finite-temperature spin-glass phase is obtained, in the absence of antiferromagnetic phase. In the spin-glass phase, the signature chaotic behavior under scale change occurs in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, yielding a piecewise chaotic interaction function.

I. ORDERING IN ONE DIMENSION: LONG-RANGE INTERACTIONS

Whereas systems with finite-range interactions do not order above zero temperature in one dimension, certain systems with long-range interactions do order.[1–6] The archetypical example is the Ising ferromagnetic models with power-law interactions, $J\,r^{-a}$. Using renormalization-group theory, we calculate finite-temperature phase diagrams and magnetization curves for these systems. We also calculate critical exponents, for the range of interaction-range exponents a.(Fig.1) Also as seen below, for antiferromagnetic interactions, the system incorporates saturated frustration and our calculation shows that the system has no antiferromagnetic ordering. For the spin-glass system, with quenched random ferromagnetic and antiferromagnetic interactions at all length scales, a spin-glass phase occurs with chaotic rescaling at all length scales and with no antiferromagnetic phase.

The model that we study is defined by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{r_1 \neq r_2} J\,|r_1 - r_2|^{-a} s_{r_1} s_{r_2} + H \sum_{r_1} s_{r_1} \qquad (1)$$

where $\beta = 1/k_B T$ is the inverse temperature, r_1 and r_2 designate the sites on the one-dimensional system, at each site r_i there is an Ising spin $s_{r_i} = \pm 1$, and the sums are over all sites in the system. For ferromagnetic and antiferromagnetic systems, the two-spin interactions J are $J = |J| > 0$ and $J = -|J| < 0$, respectively. For the

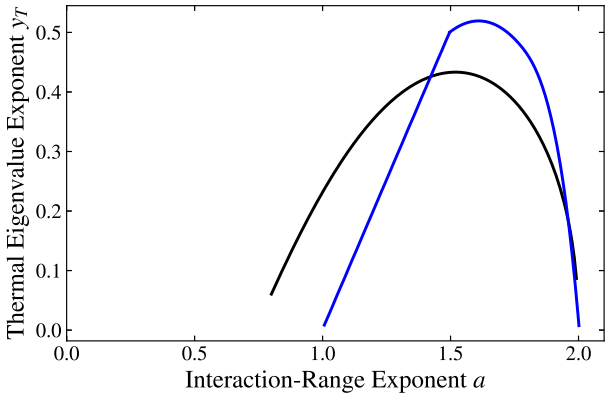


FIG. 1. Calculated thermal eigenvalue exponents y_T, as a function of the interaction-range exponent a. The curve slightly to the right is the result of Ref.[6] obtained from functional renormalization group in the local potential approximation. Two different methods, but the agreement is quite good.

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spin-glass system, for each two spins at any range, their interaction is randomly ferromagnetic (with probability $1-p$) or antiferromagnetic (with probability p). The second term in Eq. (1) is the magnetic-field H term.

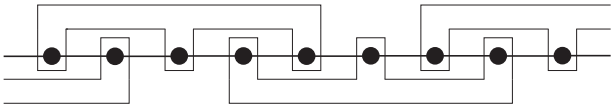


FIG. 2. Renormalization-group cells for $d = 1$. This cell structure projects both local ferromagnetism and antiferromagnetism, and therefore can also project spin-glass order.

II. METHOD: LONG-RANGE RENORMALIZATION GROUP

We solve this system with Niemeyer and van Leeuwen's two-cell cluster approximation.[7–9] The renormalization-group transformation is constructed by first choosing cells on the $d = 1$ system, as shown in Fig. 2. Each of our cells has three spins. This cell structure projects both local ferromagnetism and antiferromagnetism, and therefore can also project spin-glass order. Secondly, for each cell, a cell-spin is defined as the sign of the sum of the three spins in the cell,

$$s'_{r'} = signum(s_{r-2} + s_r + s_{r+2}) \tag{2}$$

where the signum function returns the sign of its argument, primes denote the renormalized system, and $r' = r/b$, where $b = 3$ is the length-rescaling factor of the renormalization-group transformation. The renormalized interactions are obtained from the conservation of the partition function Z,

$$Z = \sum_{\{s\}} e^{-\beta\mathcal{H}(\{s\})} = \sum_{\{s'\}}\sum_{\{\sigma\}} e^{-\beta\mathcal{H}(\{s'\},\{\sigma\})} = \sum_{\{s'\}} e^{-\beta\mathcal{H}'(\{s'\})} = Z', \tag{3}$$

where the summed variable σ represents, for each cell, the four states that give the same cell-spin value. Thus, the renormalized interactions are obtained from

$$e^{-\beta\mathcal{H}'(\{s'\})} = \sum_{\{\sigma\}} e^{-\beta\mathcal{H}(\{s'\},\{\sigma\})}. \tag{4}$$

The two-cell cluster approximation of Niemeyer and van Leeuwen consists in carrying out this transformation for two cells, including the 6 intracell interactions and the 9 intercell interactions. A recursion relation is obtained for each renormalized interaction,

$$J'_{r'} = \frac{1}{4}\ln\frac{R_{r'}(+1,+1)R_{r'}(-1,-1)}{R_{r'}(+1,-1)R_{r'}(-1,+1)}, \qquad H' = \frac{1}{4}\ln\frac{R_1(+1,+1)}{R_1(-1,-1)}, \tag{5}$$

where

$$R_{r'}(s'_0, s'_{r'}) = \sum_{\sigma_0,\sigma_{r'}} e^{-\beta\mathcal{H}_{0r'}}, \tag{6}$$

where the unrenormalized two-cell Hamiltonian contains the six intracell interactions and the 9 intercell interactions between the 6 spins in cells 0 and r'.

The different phases are determined by renormalization-group flows to different totally attractive fixed points, which are the sinks of the thermodynamic phases.[9]

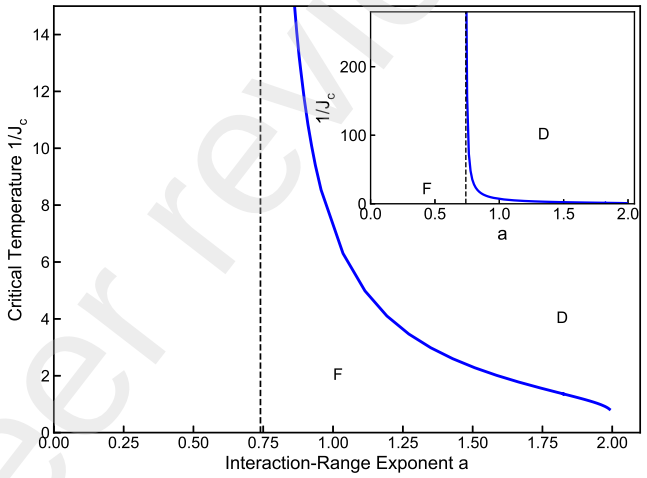


FIG. 3. Calculated phase diagram of the $d = 1$ long-range ferromagnetic Ising model with interactions $J\,r^{-a}$. Ferromagnetic (F) and disordered (D) phases are seen. A finite-temperature ferromagnetic phase transition occurs for $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, the transition becomes hybrid-order, as predicted by rigorous results [4] and also as seen from our calculated magnetization curves in Fig. 7 and our calculated critical exponents. For $a > 2$, we find that the phase transition temperature discontinuously drops to zero and there is no ordered phase above zero temperature, also as predicted by rigorous results [2, 3]. At the other end, towards the equivalent-neighbor limit, on approaching $a = 0.74$ from above, we find that the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before ($a > 0$) the neighbors become equivalent ($a = 0$), namely before the interactions become equal for all separations. To the left of the dashed line on this figure is the equivalent-neighbor regime.

III. FINITE-TEMPERATURE FERROMAGNETIC PHASE TRANSITION IN $d = 1$ BETWEEN LESS-LONG-RANGE AND EQUIVALENT-NEIGHBOR CUTOFFS

The calculated phase diagram of the $d = 1$ long-range ferromagnetic Ising model, with interactions $J\,r^{-a}$, is

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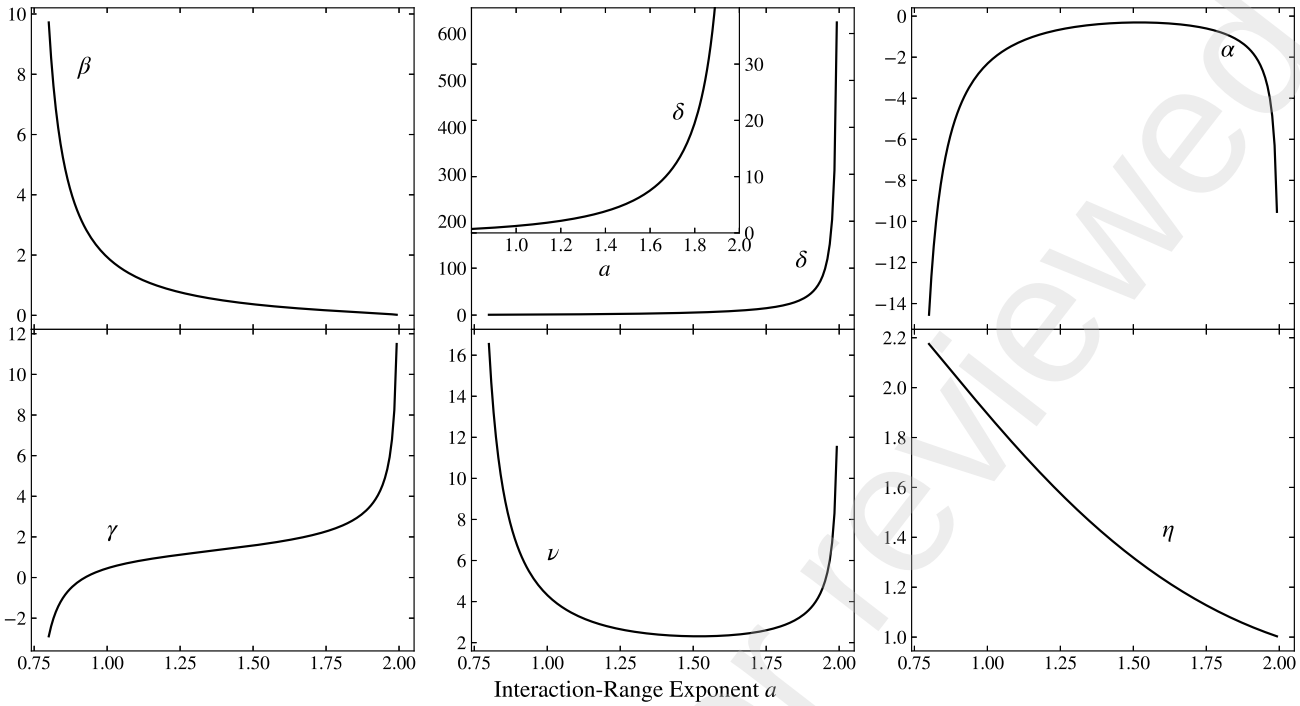


FIG. 4. Correlation-length critical exponent ν, correlation-function critical exponent η, specific heat critical exponent α, magnetization critical exponents β and δ, susceptibility critical exponent γ, as a function of interaction range a for the finite-temperature ferromagnetic phase transition. Note that β reaches 0, η reaches 1, and δ, γ diverge to infinity, as expected, as the discontinuous magnetization of the hybrid-order phase transition at $a = 2$ is reached from below. However, at $a = 2$, ν and α respectively reach plus and minus infinity, giving an unobservable essential singularity of the specific heat [11]. Thus the phase transition at $a = 2$ is hybrid order.

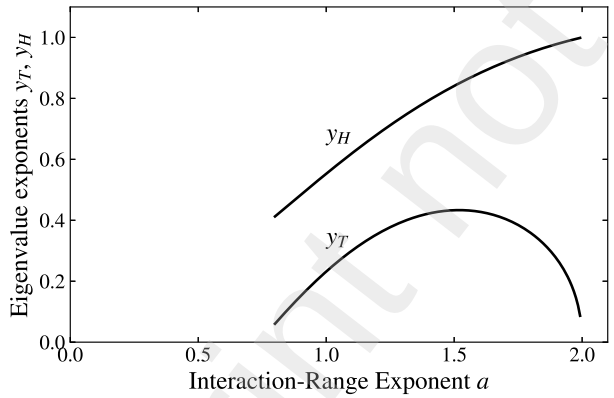


FIG. 5. Calculated thermal and magnetic eigenvalue exponents y_T and y_H, as a function of the interaction-range exponent a. These y_T and y_H respectively go to 0 and $d = 1$ as $a = 2$ is approached, respectively giving an essential singularity for the thermal thermodynamic functions and a discontinuity for the magnetization, in all meaning a hybrid-order phase transition.

shown in Fig. 3, in terms of temperature $1/J$ and interaction range a. A finite-temperature second-order ferromagnetic phase transition occurs for $0.74 < a < 2$, limits which as seen below respectively are the equivalent-neighbor cutoff and the less-long-range cutoff. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, the phase transition becomes hybrid order, as predicted by rigorous results [4] and also as seen from our calculated magnetization curves shown below and our calculated critical exponents. For $a > 2$, we find that the phase transition temperature discontinuously drops to zero and there is no ordered phase above zero temperature, also as predicted by rigorous results [2, 3]. At the other end, namely with the longest-range including infinite-range interactions, on approaching $a = 0.74$ from above, we find that the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. The interactions renormalize to infinity for all non-zero starting values. Thus, the equivalent-neighbor interactions regime is entered in fact before ($a > 0$) the neighbors become equivalent ($a = 0$), namely before the interactions become equal for all separations. Rigorous results [2, 3] show that a strong coupling regime is entered at exactly $a = 1$. Thus, our calculated value is an approximation to $a = 1$, whereas at the lesser long-range end of the ordering, the exact $a = 2$ is rendered by

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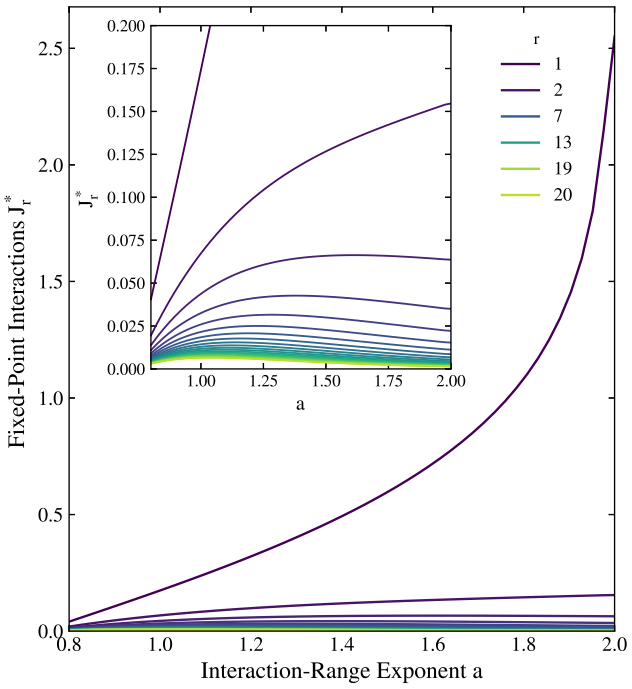


FIG. 6. Fixed-point values of the interactions J_r^*, where the subscript r is the interaction range, as a function of the interaction-range exponent a.

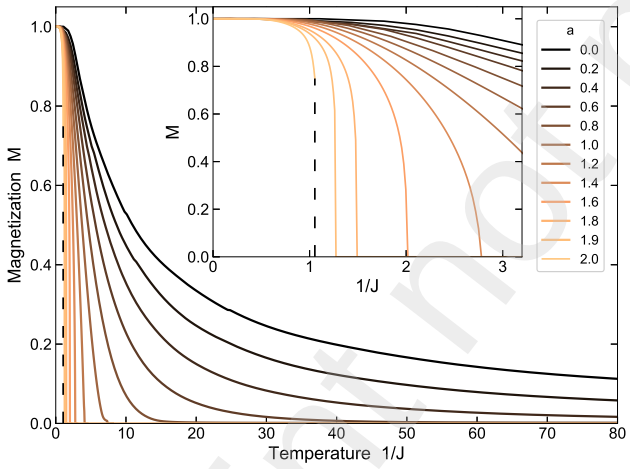


FIG. 7. Calculated magnetization curves for different values of the interaction-range exponent a. For the less-long-range threshold of $a = 2$, the dashed line shows the magnetization discontinuity, occurring only at $a = 2$. For $a < 0.74$, the magnetization decays with temperature but remains non-zero.

our calculation.

The calculated correlation-length critical exponent ν, correlation-function critical exponent η, specific heat critical exponent α, magnetization critical exponents β and δ, susceptibility critical exponent γ, continuously varying as a function of interaction-range exponent a, for the finite-temperature ferromagnetic phase transition, are shown in Fig. 4. These critical exponents are calculated, with $H = H' = 0$, from the recursion relations $J_1', ..., J_r' = \text{funct}(J_1, ..., J_r)$ of Eqs. (5,6), where the subscript r is the interaction range. Convergence is obtained by calculation up to $r = 20$. As dictated by renormalization-group theory, the critical exponents were obtained from the eigenvalues of the derivative matrix of the precisely located unstable fixed point, for each a, in the 20-dimensional space of interactions J_r. A cursory derivative of J_1 at the phase transition (which in fact is not the fixed point) does not give the correct result. The largest (and, as expected, only relevant, namely greater than 1) eigenvalue $\lambda_T = b^{y_T}$ (Figs. 1,5) of the derivative matrix (so-called recursion matrix) $\mathbf{T} = \partial J_k'/\partial J_l$ of these recursion relations at the unstable (phase-transition) fixed point $\mathbf{J_r^*}$ of the recursion relations gives the correlation-length critical exponent $\nu = 1/y_T$ and the specific heat critical exponent $\alpha = 2 - d/y_T = 2 - 1/y_T$. For each a, the 20-dimensional renormalization-group flow about the (convergedly calculated for $r = 20$ below) fixed point $\mathbf{J}^*$ of the phase transition is unstable, but $\mathbf{J}^*$ is nevertheless found precisely from the Newton-Raphson procedure,

$$\mathbf{J}^* = (\mathbf{T} - \mathbf{I})^{-1}(\mathbf{T} \cdot \mathbf{J} - \mathbf{J}'), \tag{7}$$

which is a matrix-vector product equation, where prime refers to the renormalized interaction, $\mathbf{I}$ is the identity matrix, and the recursion matrix $\mathbf{T}$ is calculated at $\mathbf{J}$. A few iterations of this procedure gives $\mathbf{J}^*$ precisely. Fig. 6 shows the calculated fixed-point interactions J_r^* as a function of the interaction-range exponent a. The eigenvalue exponents y_T and y_H are shown in Fig. 5 and respectively go to 0 and 1 as $a = 2$ is approached from below, as expected for the exponents explained below.

The magnetization critical exponents $\beta = (d - y_H)/y_T = (1 - y_H)/y_T$ and $\delta = y_H/(d - y_H) = y_H/(1 - y_H)$, the susceptibility critical exponent $\gamma = (2y_H - d)/y_T$, and the correlation-function critical exponent $\eta = 2 + d - y_H = 3 - y_H$ are calculated, at the critical fixed point $\mathbf{J}^*$, with $H = H' = 0$, from $\partial H'/\partial H = b^{y_H}$.[9] Note that at $a = 2$, the magnetization critical exponent $\beta = 0$, which is required for a first-order phase transition [10] as temperature is scanned. At $a = 2$, the other magnetization critical exponent δ and the susceptibility critical exponent γ diverge to infinity, which gives the first-order transition as the magnetic field is scanned. At $a = 2$, the correlation length critical exponent ν diverges to plus infinity and the specific heat critical exponent diverges to minus infinity (thus having an invisible essential singularity [11]), as in the $d = 2$ XY model transition [12, 13]. However, unlike the $d = 2$ XY model, the low-temperature phase has a calculated non-zero magnetization, saturating at zero temperature and going discontinuously to zero at the finite-temperature phase transition (Fig. 7), making the phase transition at $a = 2$ a hybrid-order phase transition [14].

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IV. CALCULATED MAGNETIZATION CURVES: HYBRID-ORDER PHASE TRANSITION

Magnetization curves $M(T)$ are calculated by multiplying, along the renormalization-group trajectory, the magnetization recursion relation,

$$M_0 = b^{-n} M_n \cdot \left(\frac{\partial H'}{\partial H}\right)_n \cdots \left(\frac{\partial H'}{\partial H}\right)_2 \cdot \left(\frac{\partial H'}{\partial H}\right)_1, \quad (8)$$

where the subscript n designates the nth renormalization-group transformation along the trajectory. Even when starting close to the phase transition, the trajectory takes this product close to a phase-sink fixed point. At the disordered phase sink, $M_n = 0$, and thus at the trajectory starting temperature $M_0 = 0$. At the ferromagnetic ordered phase sink, $M_n = 1$ and $(\partial H'/\partial H)_n = b$, and thus at the starting temperature non-zero magnetization is obtained.

The thus calculated magnetization curves for the entire range of temperatures for each a are given in Fig. 7. For $0.74 < a < 2$, the magnetization curves show a second-order phase transition, with the magnetizations continuously reaching zero as the critical temperature is approached from below. The second-order phase transition temperature monotonically decreases between these two limits of a. At $a = 2$, the phase transition becomes hybrid order, as seen by the large discontinuity in calculated magnetization only at $a = 2$. For $a > 2$, we find that the phase transition temperature discontinuously drops to zero and there is no ordered phase above zero temperature, as also predicted by rigorous results [2, 3]. At the other end, on approaching $a = a_{eq} = 0.74$ from above, we find that the phase transition temperature diverges to infinity. For $a < 1.16$, the magnetization critical exponent $\beta > 1$ and interestingly the magnetization reaches zero temperature horizontally, in preparation for the $a < a_{eq}$ equivalent-neighbor regime. For $a < a_{eq}$, the calculated magnetization is non-zero at all non-infinite temperatures as seen Fig. 7, and the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered in fact before $(a > 0)$ the neighbors become equivalent $(a = 0)$, namely before the interactions become equal for all separations. To the left of the dashed line on this figure is the equivalent-neighbor regime.

V. FINITE-TEMPERATURE SPIN-GLASS PHASE IN $d = 1$ WITHOUT ANTIFERROMAGNETIC PHASE

The antiferromagnetic, overly frustrated without randomness, system does not have a finite-temperature phase transition, as all renormalization-group trajectories flow to the disordered phase sink fixed pont. This is also proven rigorously.[15] However, the spin-glass system, where all couplings for all separations are randomly ferromagnetic (with probability $1 - p$) or antiferromagnetic (with probability p), does have finite-temperature spin-glass phase transitions and chaos inside the spin-glass phase, as seen in Fig. 8.

The essence of a spin-glass phase is chaos under scale change, as the renormalization-group attractor, namely phase sink.[17–20] This is clearly seen in the lower portion of Fig. 8, as the interactions for all separations become chaotic under renormalization-group transformations, after three renormalization-group transformations.

Thus, the spin-glass phase shows its signature of chaos under rescaling [17–20], but in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, as seen in the lower panel of Fig. 8, yielding an interaction potential that is piecewise chaotic.

This truly unusual spin-glass phase diagram, actually does not have an antiferromagnetic phase but has a spin-glass phase. Nevertheless, typical spin-glass system reentrance [16] is seen in this phase diagram, where as temperature is lowered at fixed antiferromagnetic bond concentration p, the ferromagnetic phase appears, but disappears at further lower temperature.

For a previous $d = 1$ Ising spin-glass study, with short-range interactions and a zero-temperature spin-glass phase, see [21].

VI. CONCLUSION

We have solved the $d = 1$ Ising ferromagnet, antiferromagnet, and spin glass with long-range power-law interactions $J\,r^{-a}$, for all interaction-range exponents a by a renormalization-group transformation that simultaneously projects local ferromagnetism, antiferromagnetism, and spin-glass order. In the ferromagnetic case, $J > 0$, a finite-temperature second-order ferromagnetic phase occurs for interaction range $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. The critical exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$ for the second-order phase transitions are calculated, from a large recursion matrix, varying as a function of a. At $a = 2$, we find that the phase transition becomes hybrid order, as predicted by rigorous results (hybrid order, also considering the essential singularity found in the specific heat). For $a > 2$, we find that the phase transition temperature discontinuously drops to zero and for $a > 2$ there is no ordered phase above zero temperature, as also predicted by rigorous results. At the other end, towards the equivalent-neighbor limit, on approaching $a = 0.74$ from above, namely increasing the range of the interaction, we find that the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before $(a > 0)$ the neighbors become equivalent, namely before the interactions become equal $(a = 0)$ for all separations.

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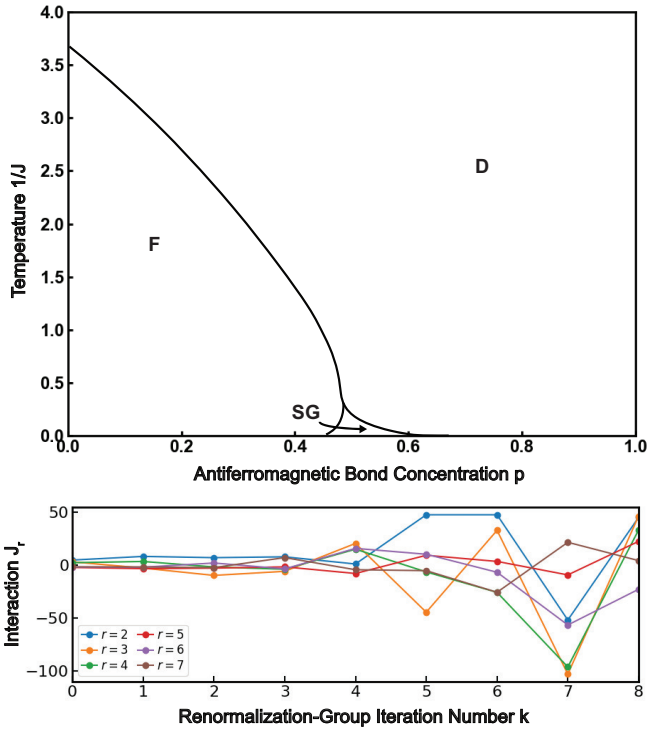


FIG. 8. Calculated finite-temperature phase diagram of the $d = 1$ long-range Ising spin-glass system with interaction-range exponent $a = 1$, where all couplings for all separations are randomly ferromagnetic (with probability $1 - p$) or antiferromagnetic (with probability p). Ferromagnetic (F), spin-glass (SG), and disordered (D) phases are seen. This truly unusual spin-glass phase diagram, actually does not have an antiferromagnetic phase but has a spin-glass phase. Bottom panel: Chaos inside the spin-glass phase in $d = 1$. The spin-glass phase shows the chaos under rescaling signature [17–20], in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, as seen in the lower panel of this figure, yielding a piecewise chaotic interaction potential.

For the antiferromagnetic case, $J < 0$, all triplets of spins at all ranges have competing interactions and this highly frustrated system does not have an ordered phase.

In the spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p), a finite-temperatures spin-glass phase is obtained, in the absence of antiferromagnetic phase. A truly unusual phase diagram, with reentrance around the ferromagnetic phase, is obtained. In the spin-glass phase, the signature chaotic behavior under scale change occurs in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, yielding a piecewise chaotic interaction function.

ACKNOWLEDGMENTS

Support by the Academy of Sciences of Turkey (TÜBA) is gratefully acknowledged.

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